On the Aristotelian Square of Opposition

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Abstract

A common misunderstanding is that there is something logically amiss with the classical square of opposition, and that the problem is related to Aristotle’s and medieval philosophers’ rejection of empty terms. But Parsons 2004 convincingly shows that most of these philosophers did not in fact reject empty terms, and that, when properly understood, there are no logical problems with the classical square. Instead, the classical square, compared to its modern version, raises the issue of the existential import of words like all; a semantic issue. I argue that the modern square is more interesting than Parsons allows, because it presents, in contrast with the classical square, notions of negation that are ubiquitous in natural languages. This is an indirect logical argument against interpreting all with existential import. I also discuss some linguistic matters bearing on the latter issue.

1 The Classical Square

When Aristotle invented the very idea of logic some two thousand four hundred years ago, he focused on the analysis of quantification. Operators like and or were added later (by Stoic philosophers). Aristotle’s syllogisms can be seen as a formal rendering of certain inferential properties, hence of aspects of the meaning, of the expressions all, some, no, not all. The logical properties of these quantifiers were expressed in two ways:

• the particular inference forms that Aristotle called syllogisms;
• certain other logical relations that later were illustrated in the so-called square of opposition.

A syllogism has the form:\footnote{This is the so-called first figure – three more figures are obtained by permuting $AB$ or $BC$ in the premisses. This gives 256 possible syllogisms.}

\[
Q_1AB \\
Q_2BC \\
Q_3AC
\]
where each of $Q_1, Q_2, Q_3$ is one of the four expressions above. Some syllogisms are valid, whereas most of the possible syllogisms are invalid – Aristotle defined these notions in a way that leaves little to be desired from a modern point of view. Moreover, he performed the remarkable meta-logical feat of systematizing all the valid ones, choosing a few as axioms and deriving the rest with given inference rules.\footnote{Essentially, he proved a completeness theorem for (valid) syllogisms. Not only was this unprecedented – logic did not exist before – but such a technical perspective on logic was not to reappear until well into the 20th century (with the first proofs of the completeness of the propositional calculus). On this matter, see \cite{Corcoran2003} for an interesting comparison between Aristotle and Boole.}

Note the use of variables, which emphasizes that the syllogisms are inference schemes, and furthermore significantly facilitates the formulation and study of these schemes. Aristotle was the first to use variables for linguistic expressions, in this case names of properties, or as they are usually called in traditional logic, terms. $\text{allAB}$ is read \textit{All As are B}, $\text{noAB}$ is \textit{No As are B}, etc.

The square of opposition expresses another kind of logical 'laws', mostly to do with ways in which basic propositions with the four quantifiers contradict or 'oppose' each other. The classical square, i.e., the square as it appears in the work of Aristotle (though he did not use the diagram\footnote{Apparently, the first to use a diagrammatic representation was Apuleios of Madaura (2nd century A.D.)}) and in most subsequent work up to the advent of modern logic in the late 19th century, is as in Figure 1.

![Figure 1: The Classical Square](image-url)

The $\text{A}$ and $\text{E}$ quantifiers are called \textit{universal}, whereas the $\text{I}$ and $\text{O}$ quantifiers are \textit{particular}. Also, the $\text{A}$ and $\text{I}$ quantifiers are called \textit{affirmative}, and the $\text{E}$ and $\text{O}$ quantifiers \textit{negative}. An important point is that the quantifier in the $\text{A}$ position is what I have here called $\text{all}_{ei}$, that is, the quantifier \textit{all} with \textit{existential import}. So $\text{all}_{ei}(A, B)$ in effect means that all $A$s are $B$s and there are some $A$s. This is explicit with many medieval authors, but also clearly implicit in Aristotle's
work, for example, in the fact that he (and almost everyone else doing syllogistics before the age of modern logic) considered the following scheme as valid:

\[(1)\]  
\[
\begin{align*}
&\text{all } AB \\
&\text{all } BC \\
&\text{some } AC
\end{align*}
\]

The logical relations in the classical square are as follows: Diagonals connect *contradictory* propositions, i.e., propositions that cannot have the same truth value. The *A* and *E* propositions are *contrary*: they cannot both be true (note that this too presupposes that the *A* quantifier has existential import). The *I* and *O* propositions are *subcontrary*: they cannot both be false. Finally, the *E* proposition is *subalternate* to the *A* proposition: it cannot be false if the *A* proposition is true; in other words, it is *implied* by the *A* proposition (again showing that the *A* quantifier was taken to have existential import). Similarly for the *O* and *E* propositions.

In addition, the ‘convertibility’ of the *I* and *E* positions, i.e., the fact that no *As* are *B* implies that no *Bs* are *A*, and similarly for *some*, was also taken by Aristotle and his followers to belong to the basic logical facts about the square of opposition.

Notice that it follows that the *A* and *O* propositions are negations of each other (similarly for the *I* and *E* propositions). Thus, the quantifier at the *O* position means that either something in *A* is not in *B*, or there is nothing in *A*. So the *O* proposition is *true* when *A* is empty, i.e., contrary to the modern (logical) usage, the quantifier *not all* does not have existential import. *(Q has existential import if Q(A, B) implies that A is non-empty.)* Indeed, the usual classical opinion was that affirmative quantifiers, but not negative ones, have existential import.

But it seems that during the late 19th and 20th centuries this fact was usually forgotten, and consequently it was thought that the logical laws described by the classical square of opposition were deficient or even inconsistent.\(^4\) Furthermore, it was often supposed that the problem arose from insufficient clarity about empty terms, i.e., expressions denoting the empty set.\(^5\)

For a detailed and convincing argument that most of this later discussion simply rests on a mistaken interpretation of the classical square, I refer to [Parsons 2004]. The upshot is that, apparently, neither Aristotle nor (with a few exceptions) medieval philosophers disallowed empty terms, and some medieval philosophers explicitly endorsed them. Here is an example. In Paul of Venice’s important opus *Logica Magna* (ca 1400), he gives

\[(2)\]  
\[
\text{Some man who is a donkey is not a donkey}
\]

\(^4\)For an example of this, cf. [Kneale & Kneale 1962], pp. 55–60.

\(^5\)Consequently it is often assumed that the problems go away if one restricts attention to non-empty terms. But actually one has to disallow their complements, i.e., universal terms, as well, which seems less palatable. In any case, neither restriction is motivated, as we will see.
as an example of sentence which is true since the subject term is empty (see [Parsons 2004], section 5). So he allows empty terms, and confirms the interpretation of the $O$ quantifier just mentioned (reading not all as some . . . not).

In fact, as long as one remembers that the $O$ quantifier is the negation of the quantifier $\text{all}_{ei}$, nothing is wrong with the logic of the classical square of opposition.

2 The Modern Square

A totally different issue, however, is which interpretation of words like all and every is ‘correct’, or rather, most adequate for linguistic and logical purposes. Nowadays, all is used without existential import, and the modern square of opposition is as in Figure 2.

![Figure 2: The Modern Square](image)

[Parsons 2004] appears to think this square is impoverished and less interesting, but I disagree on that point. The main virtue of the modern square is that it depicts important forms of negation that appear in natural (and logical) languages.
negations and duals

As in the classical square, the diagonals indicate 'contradictory', or outer negation. When \( Q_i \) and \( Q_j \) are at the ends of a diagonal, the proposition \( A's \ are \ B \) is simply the negation of \( Q_j \ A's \ are \ B \), i.e., it is equivalent to It is not the case that \( Q_j \ A's \ are \ B \). This propositional negation 'lifts' to the (outer) negation of a quantifier, and we can write \( Q_i = \neg Q_j \) (and hence \( Q_j = \neg \neg Q_i = \neg Q_i \)).

A horizontal line between \( Q_i \) and \( Q_j \) now stands for what is often called inner negation (or post-complement): here \( Q_i \ A's \ are \ B \) is equivalent to \( Q_j \ A's \ are \ not \ B \), which can be thought of as applying the inner negation \( Q_j \neg \) to the denotations of \( A \) and \( B \).

Finally, a vertical line in the square indicates that the respective quantifiers are each other's duals, where the dual of \( Q_i \) is the outer negation of its inner negation (or vice versa): \( Q_i^d = \neg (Q_i \neg) = (\neg Q_i) \neg = \neg Q_i \neg \).

The modern square is closed under these forms of negation: applying any number of these operations to a quantifier in the square will not lead outside it. For example, \((no^2) \neg = \neg no \neg \neg = \neg no = some\).

Each of these forms of negation has natural manifestations in real languages. Moreover, the modern square of opposition is by no means limited to the quantifiers discussed so far.

To see this, note that Aristotle’s notion of a quantifier is – in modern terms – on the syntactic side essentially that of a binary relation (symbol) between terms, and thus on the semantic side a corresponding relation between the denotation of terms, i.e. a binary relation between sets:

\[
\begin{align*}
all(A, B) & \iff A \subseteq B \\
not all(A, B) & \iff A - B \neq \emptyset \\
all_{ci}(A, B) & \iff A \subseteq B \land A \neq \emptyset \\
not all_{ci}(A, B) & \iff A - B \neq \emptyset \lor A = \emptyset \\
some(A, B) & \iff A \cap B \neq \emptyset \\
no(A, B) & \iff A \cap B = \emptyset 
\end{align*}
\]

Now many languages have an essentially unlimited class of similar expressions – determiners (Det) – that syntactically combine with nouns (N) to form noun phrases (NP),

\[
(3)
\]

\[
S \\
NP \\
Det \\
most \\
N \\
students \\
\]

\[
VP \\
smoke
\]
and semantically denote binary relations between sets (or type \(\langle 1, 1 \rangle\) generalized quantifiers, as they are nowadays called)\(^6\):

- **at least two** \((A, B) \iff |A \cap B| \geq 2\)
- **exactly five** \((A, B) \iff |A \cap B| = 5\)
- **all but three** \((A, B) \iff |A - B| = 3\)
- **more than two thirds of the** \((A, B) \iff |A \cap B| > \frac{2}{3} \cdot |A|\)
- **most** \((A, B) \iff |A \cap B| > |A - B|\)
- **the ten** \((A, B) \iff |A| = 10 \text{ and } A \subseteq B\)
- **John's** \((A, B) \iff \emptyset \neq A \cap \{a : \text{John 'possesses' } a\} \subseteq B\)
- **some but not all** \((A, B) \iff A \cap B \neq \emptyset \neq A - B\)
- **infinitely many** \((A, B) \iff A \cap B \text{ is infinite}\)
- **an even number of** \((A, B) \iff |A \cap B| \text{ is even}\)

A sentence of the form (3) can be negated by putting it is not the case that in front, or by negating the VP, or by doing both. Sometimes these negations can be effected by choosing another Det. For example, applying the first kind of negation to Some students passed we get It is not the case that some students passed, but this can also be expressed by No students passed (outer negation). In the second case we obtain Some students did not pass, and this could instead be put Not all students passed (inner negation).

Thus, the inner and outer negation as well as the dual of a determiner denotation is sometimes also a determiner denotation:

- \(\neg \text{some} = \text{no}; \text{some} \neg = \text{not all}; \text{some}^d = \text{all}\)
- \(\neg \text{more than half of the} = \text{at most half of the}; \text{more than half of the} \neg = \text{less than half of the}; \text{more than half of the}^d = \text{at least than half of the}\)

But regardless of whether a quantifier \(Q\) is the denotation of some determiner, it always spans a corresponding square of opposition:

- \(\text{square}(Q) = \{Q, \neg Q, Q \neg, Q^d\}\)

Here are some easily verified facts about squares. The **trivial** quantifiers 0 and 1 are the empty and the universal relations between sets, respectively. \(Q\) is **non-trivial** if it is different from these two.

\(^6\)More precisely, on each universe they denote such relations. Here we assume a fixed (discourse) universe in the background. For an account of how the logical theory of generalized quantifiers applies to natural language semantics, see, for example, [Keenan and Westerståhl 1997].
2.1 Fact
(a) \(\text{square}(0) = \text{square}(1) = \{0, 1\}\).
(b) If \(Q\) is non-trivial, so are the other quantifiers in its square.
(c) Each quantifier in a square spans that same square. That is, if \(Q' \in \text{square}(Q)\), then \(\text{square}(Q) = \text{square}(Q')\). So any two squares are either identical or disjoint.
(d) \(\text{square}(Q)\) has either two or four members.

The difference between the classical Aristotelian square and its modern version might at first seem rather insignificant: all instead of \(\text{all}_e\), and similarly for not all. But we now see that the principled differences are huge. First, whereas outer negation is presented in both squares, neither inner negation nor dual is contained in the classical square. For example, the dual of the quantifier \(\text{all}_e\) is the quantifier which holds of \(A\) and \(B\) iff either some \(A\) is \(B\) or \(A\) is empty. The latter quantifier is rather ‘unnatural’; and doesn’t seem to be denoted by any determiner.

Second, the inferential relations along the sides of the classical square are not present in the modern square. In general, a quantified statement neither implies nor is implied by the dual statement, for example. And a quantified statement and its inner negated form may both be true, so they are not contraries in the classical sense.

Third, the classical square is not generated by any of its members. To make this claim precise, let us define a classical square as an arrangement of four quantifiers as in Figure 1 and with the same logical relations – contradictories, contraries, subcontraries, and subalternates – holding between the respective positions. Then each position will determine the quantifier at the diagonally opposed position, i.e, its outer negation, but not the quantifiers at the other two positions. For example, the following fact holds:

2.2 Fact
For every \(n \geq 1\), the square

\[\begin{array}{cccc}
\text{A: at least } n & \text{E: no} & \text{I: some} & \text{O: fewer than } n \\
\end{array}\]

is classical. More generally, for \(n \geq k\),

\[\begin{array}{cccc}
\text{A: at least } n & \text{E: fewer than } k & \text{I: at least } k & \text{O: fewer than } n \\
\end{array}\]

is classical.

Summing up, the contrast between the classical and the modern square of opposition concerns both logic and semantics. Though each square is coherent, the logical relations they present are quite different (one may dispute which group of relations is more ‘interesting’). The main semantic issue at stake is not whether empty terms should be allowed or not, but whether a statement of the form \(\text{All } A\ \text{as are } B\) can be \textit{true} when \(A\) is an empty term. That is, the issue is whether \(\text{all}\) and its cognates have existential import or not.\footnote{It goes without saying that my defense of the modern square of opposition contains no...}
3 Existential Import

Does all have existential import? And what kind of question is this?

Everyone is familiar with the fact that it is usually strange to assert that all As are B when one knows there are no As. Surely a main question is whether the existential import that is often ‘felt’ with uses of all belongs to the meaning – the truth conditions – or rather is a presupposition or a Gricean implicature.

This is to a large extent an empirical matter, but perhaps not entirely so. There is no unanimity among linguists or philosophers of language about where the line between semantics and pragmatics should be drawn, and in doubtful cases, other considerations could play a role too. Concerning all, a rather strong argument in favor of the interpretation without existential import was given in the previous section, or so it seems to me. Only that quantifier fits into the modern square of opposition, and thus has those very simple and unique ties to some, no, and not all, in terms of the three kinds of negation that any semantics of natural languages has to account for anyway. In short, logical simplicity, generality, and coherence speak in favor of the modern interpretation.

Indeed, it is clear that in logic, the use allei in place of all – though in principle possible without change of expressive power – would have no advantages but only complicate things.8

But what about natural language? A semanticist is not free to stipulate a meaning for a word just because it is logically simpler than an alternative – the alternative might still be the speakers’ choice, and if this is clearly so, the speakers rule.

However, the data concerning all is not crystal clear. I will end by mentioning at least some of the relevant facts.

As I said, care must be taken to distinguish the fact that if one knows that the noun A has empty denotation, it would often be odd to utter a sentence of the form Q A’s are B, from facts about the perceived falsity (or truth) of the sentence in this case. So one way to be careful is to use sentences where it clearly can be unknown whether the noun denotation is empty or not, such as the following:

Note that if allei x (Px, Rx) means ∀x(Px → Rx) ∧ ∃xPx, then all ei x (Px, Px) will be equivalent to ∃xPx. So ∃, and hence ∀ and all, are expressible by means of all ei (and propositional connectives), though there would be no advantage of expressing them in this roundabout way.

Note also that the standard ∀ does have a kind of existential import, in that ∀x Px is logically equivalent to ∀xPx ∧ ∃xPx, due to the usual assumption the the universe is non-empty. This too is practical for many purposes, but not for all: in the context of relativizing a formula to a smaller universe it is convenient to allow that universe to be empty, and relativization turns out to be highly relevant to natural language quantification; see e.g. [Keenan and Westerståhl 1997], p. 858. In any case, the existential import of ∀ in the above sense has nothing to do with whether the determiner all has existential import or not.
(4) All solutions to this system of equations are integers.

This sentence could be acceptable (for example, provable in a certain theory) regardless of whether there were any solutions at all to the system of equations in question, and certainly regardless of whether one has any knowledge about the existence of such solutions.

Similarly for statements of laws or rules:

(5) All trespassers will be prosecuted.

There doesn’t have to be any actual trespassers for this to hold.

Another strategy is to consider whether an eventual presupposition or implicature of non-emptiness can be explicitly canceled. This strategy gives some definite results. First, a test case:

(6) # It is true that at least two graduate students at the party were drunk, because there were in fact no graduate students at the party.

Speakers would not accept this: they would claim that the second part of the sentence contradicts the first. This is fortunate, since it shows that at least two does have existential import – which we of course knew anyway.

Next, a fairly clear negative case:

(7) It is true that no graduate students at the party were drunk, because there were in fact no graduate students at the party.

From the first part of the sentence, one would normally assume that there were graduate students at the party. The second part contradicts this assumption. But the first part doesn’t become false with the cancellation. It remains true, if a bit odd to say. We may conclude that existential import with no is an implicature (or something similar), but not part of its meaning.

Now consider

(8) It is true that all graduate students at the party were drunk, because there were in fact no graduate students at the party.

Again, it seems that the claim that there were graduate students at the party is cancelable, though perhaps with slightly more difficulty than in the previous case. But imagine the following dialogue:

- All graduate students at the party were drunk.
- I’m glad I didn’t go!
- But nobody was drunk at that party.
- But you just said there were drunk people there!
- No I didn’t say that; I only said that all graduate students at the party were drunk, which happens to be true because there were no graduate students at the party!

The first speaker is seriously misleading the second. But the reason he can do this, and thus be judged uncooperative or even devious while still not being incoherent, is precisely that implying something is not the same as saying it.
At least, that is a common verdict. But the issue is somewhat subtle. To appreciate this, compare with the following:

(9) a. # It is true that the graduate students at the party were drunk, because there were in fact no graduate students at the party.

b. # It is true that Henry’s graduate students were drunk, because Henry doesn’t have any graduate students.

Most semanticists find these incoherent enough to conclude that the and Henry’s do have existential import, in contrast with all.

Perhaps one can sum up the situation as follows. Assertions of universal statements have varying degrees of existential import. An assertion needs some sort of warrant. Sometimes the warrant has no information about about the emptiness or not of the first argument of the determiner (the restriction term), as in (5), or is explicitly neutral about its emptiness, as in (4). But these cases are a bit special. Usually, the warrant is some observation or inference, and then the assertion can imply rather strongly that the restriction argument is non-empty. But note that all of these remarks apply to assertions. If one thinks of the linguistic meaning of an expression as, roughly, what is common to all assertions involving that expression (its ‘assertion potential’), it makes sense not to endow all with existential import. The existential import of assertions of universal statements is rather a matter for pragmatics than for semantics.

References


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9The border-line between semantics and pragmatics, and in particular the notion of what is said – as opposed to, for example, what is implicated – by an utterance of a sentence is the subject of a long debate in the philosophy of language, a debate which has been intense in recent years. One’s interpretation of the above examples may depend on one’s position in that debate. For an up-to-date overview (and one particular position), see [Recanati 2004].